

# Continuum-wise expansiveness for generic diffeomorphisms

Manseob Lee

*Department of Mathematics, Mokwon University,  
Daejeon, 302-729, Korea.*

*E-mail: lmsds@mokwon.ac.kr.*

## Abstract

Let  $M$  be a closed smooth manifold and let  $f : M \rightarrow M$  be a diffeomorphism.  $C^1$ -generically, a continuum-wise expansive satisfies Axiom A without cycles. Let  $M = \mathbb{T}^3$  and let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ . There are a  $C^1$  neighborhood  $\mathcal{U}(f)$  of  $f \in \mathcal{RT}(\mathbb{T}^3)$  and a residual set  $\mathcal{R} \subset \mathcal{U}(f)$  such that for any  $g \in \mathcal{R}$ ,  $g$  is not continuum-wise expansive, where  $\mathcal{RT}(\mathbb{T}^3)$  is the set of all robustly transitive diffeomorphisms on  $\mathbb{T}^3$ .

## 1 Introduction

Let  $M$  be a closed smooth manifold with  $\dim M \geq 2$ , and let  $\text{Diff}(M)$  be the space of diffeomorphisms of  $M$  endowed with the  $C^1$  topology. Denote by  $d$  the distance on  $M$  induced from a Riemannian metric  $\|\cdot\|$  on the tangent bundle  $TM$ . In differentiable dynamical systems, expansiveness is a very useful notion to investigate for stability theory. For instance, Mañé [16] proved that the  $C^1$ -interior of the set of expansive diffeomorphisms coincides with the set of quasi-Anosov diffeomorphisms. Here  $f$  is *quasi-Anosov* if for all  $v \in TM \setminus \{0\}$ , the set  $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$  is unbounded. Let  $f \in \text{Diff}(M)$ . We say that  $f$  is *expansive* if there is  $e > 0$  such that for any  $x, y \in M$  if  $d(f^i(x), f^i(y)) < e$  for all  $i \in \mathbb{Z}$  then  $x = y$ . Denote by  $\mathcal{E}$  the set of all expansive diffeomorphisms. From now, we introduce various expansiveness (N-expansive, countably expansive, measure expansive [18, 19]) which are general notions of original expansiveness.

---

<sup>1</sup> 2010 *Mathematical Subject Classification*: 37D30; 34D10.

*Key words and phrases*: expansive; continuum-wise expansive; Axiom A; partially hyperbolic; generic.

We say that  $f$  is  $N$ -*expansive* if there is  $e > 0$  such that for any  $x \in M$ , the number of elements of the set  $\Gamma_e(x) = \{y \in M : d(f^i(x), f^i(y)) < e \text{ for all } i \in \mathbb{Z}\}$  is less than  $N$ . Denote by  $\mathcal{GE}$  the set of all  $N$ -expansive diffeomorphisms on  $M$ . We say that  $f$  is *countably expansive* if there is  $e > 0$  such that for  $x \in M$ , the number of elements of the set  $\Gamma_e(x) = \{y \in M : d(f^i(x), f^i(y)) < e \text{ for all } i \in \mathbb{Z}\}$  is countable, where  $e$  is an expansive constant for  $f$ .

Note that if a diffeomorphism  $f$  is expansive then  $\Gamma_e(x) = \{x\}$  for  $x \in M$ . Thus if a diffeomorphism  $f$  is expansive then  $f$  is countably expansive, but the converse is not true (see [19]).

For a Borel probability measure  $\mu$  on  $M$ , we say that  $f$  is  $\mu$ -expansive if there is  $\delta > 0$  such that  $\mu(\Gamma_\delta(x)) = 0$  for all  $x \in M$ . In this case, we say that  $\mu$  is a *expansive measure* for  $f$ . We say that  $f$  is *measure expansive* if it is  $\mu$ -expansive for every non-atomic Borel probability measure  $\mu$ . Denote by  $\mathcal{ME}$  the set of all measure-expansive diffeomorphisms on  $M$ .

Continuum-wise expansive diffeomorphisms was introduced by Kato [12]. A set  $\Lambda$  is *nondegenerate* if the set  $\Lambda$  is not reduced to one point. We say that  $\Lambda \subset M$  is a *subcontinuum* if it is a compact connected nondegenerate subset of  $M$ .

**Definition 1.1** *A diffeomorphism  $f$  on  $M$  is said to be continuum-wise expansive if there is a constant  $e > 0$  such that for any nondegenerate subcontinuum  $\Lambda$  there is an integer  $n = n(\Lambda)$  such that  $\text{diam} f^n(\Lambda) \geq e$ , where  $\text{diam} \Lambda = \sup\{d(x, y) : x, y \in \Lambda\}$  for any subset  $\Lambda \subset M$ . Such a constant  $e$  is called a continuum-wise expansive constant for  $f$ .*

Note that every expansive diffeomorphism is continuum-wise expansive diffeomorphism, but its converse is not true (see [12, Example 3.5]). Denote by  $\mathcal{CWE}$  the set of all continuum-wise expansive diffeomorphisms of  $M$ . In [4], Artigue showed that

$$\mathcal{E} \Rightarrow \mathcal{GE} \Rightarrow \mathcal{CE} = \mathcal{ME} \Rightarrow \mathcal{CWE},$$

where  $\mathcal{CE}$  is the set of all countably expansive diffeomorphisms on  $M$ . For a  $C^1$  perturbation expansive diffeomorphism, we can find the following result (see [5, 13, 21, 22]). Denote by  $\text{int}A$  the  $C^1$ -interior of a set  $A$  of  $C^1$ -diffeomorphisms of  $M$ .

**Theorem 1.2** *Let  $f \in \text{Diff}(M)$ . Then we have the following*

$$\text{int}\mathcal{E} = \text{int}\mathcal{GE} = \text{int}\mathcal{ME} = \text{int}\mathcal{CWE}.$$

Let  $\Lambda$  be a closed  $f$ -invariant set. We say that  $\Lambda$  is *hyperbolic* if the tangent bundle  $T_\Lambda M$  has a  $Df$ -invariant splitting  $E^s \oplus E^u$  and there exists constants  $C > 0$  and  $0 < \lambda < 1$  such that

$$\|D_x f^n|_{E_x^s}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E_x^u}\| \leq C\lambda^n$$

for all  $x \in \Lambda$  and  $n \geq 0$ . If  $\Lambda = M$  then  $f$  is said to be Anosov.

It is well known that if a diffeomorphism  $f$  is Anosov then it is quasi-Anosov, but the converse is not true (see [10]). Thus if a diffeomorphism  $f$  is Anosov then  $f$  is expansive, N-expansive, measure expansive, countably expansive and continuum-wise expansive. We say that  $f$  satisfies *Axiom A* if the non-wandering set  $\Omega(f)$  is hyperbolic and it is the closure of  $P(f)$ . A point  $x \in M$  is said to be *non-wandering* for  $f$  if for any non-empty open set  $U$  of  $x$  there is  $n \geq 0$  such that  $f^n(U) \cap U \neq \emptyset$ . Denote by  $\Omega(f)$  the set of all non-wandering points of  $f$ . It is clear that  $\overline{P(f)} \subset \Omega(f)$ . A diffeomorphism  $f$  is  $\Omega$ -stable if there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that for any  $g \in \mathcal{U}(f)$  there is a homeomorphism  $h : \Omega(f) \rightarrow \Omega(g)$  such that  $h \circ f = g \circ h$ , where  $\Omega(g)$  is the non-wandering set of  $g$ . A subset  $\mathcal{G} \subset \text{Diff}(M)$  is called *residual* if it contains a countable intersection of open and dense subsets of  $\text{Diff}(M)$ . A dynamic property is called  $C^1$  generic if it holds in a residual subset of  $\text{Diff}(M)$ . Arbieto [3] proved that if a  $C^1$  generic diffeomorphism  $f$  is expansive then it is Axiom A without cycles. Lee [13] proved that if a  $C^1$  generic diffeomorphism  $f$  is N-expansive then it is Axiom A without cycles. Very recently, Lee [15] proved that if a  $C^1$  generic diffeomorphism  $f$  is measure expansive then it is Axiom A without cycles. From that, we consider  $C^1$  generic continuum-wise expansive diffeomorphisms. The following is a main result.

**Theorem A** *For  $C^1$  generic  $f \in \text{Diff}(M)$ , if  $f$  is continuum-wise expansive then it is Axiom A without cycles.*

We say that a  $f$ -invariant closed set  $\Lambda$  admits a *dominated splitting* if the tangent bundle  $T_\Lambda M$  has a continuous  $Df$ -invariant splitting  $E \oplus F$  and there exist constants  $C > 0$  and  $0 < \lambda < 1$  such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all  $x \in \Lambda$  and  $n \geq 0$ . If the dominated splitting can be written as a sum

$$T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_i \oplus E_{i+1} \oplus \cdots \oplus E_k,$$

then we say that the sum is *dominated* if for all  $i$  the sum

$$(E_1 \oplus E_2 \oplus \cdots \oplus E_i) \oplus (E_{i+1} \oplus E_{i+2} \oplus \cdots \oplus E_k)$$

is dominated. Note that the decomposition is called the *finest* dominated splitting if we can't decompose in a non-trivial way subbundle  $E_i$  appearing in the splitting.

The set  $\Lambda$  is *partially hyperbolic* if there is a dominated splitting  $E \oplus F$  of  $T_\Lambda M$  such that either  $E$  is contracting or  $F$  is expanding.

**Definition 1.3** We say that a compact  $f$ -invariant set  $\Lambda \subset M$  is strongly partially hyperbolic if the tangent bundle  $T_\Lambda M$  has a dominated splitting  $E^s \oplus E^c \oplus E^u$  and there exist  $C > 0$  and  $0 < \lambda < 1$  such that for all  $v \in E^s$ , we have  $\|Df^n(v)\| \leq C\lambda^n\|v\|$  for all  $n \geq 0$ , and for all  $v \in E^u$ , we have  $\|Df^{-n}(v)\| \leq C\lambda^n\|v\|$  for all  $n \geq 0$ , where  $E^c$  is the central direction of the splitting.

Note that if  $\Lambda$  is hyperbolic for  $f$  then it is strongly partially hyperbolic and  $E^c$  is not empty, that is,  $E^c = \{0\}$ . For a partially hyperbolic diffeomorphism, Burns and Wilkinson [6] showed the following lemma.

**Lemma 1.4** Let  $\Lambda$  be a compact  $f$ -invariant set with a partially hyperbolic splitting,

$$T_\Lambda M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u.$$

Let  $E^{cs,i} = E^s \oplus E_1^c \oplus \cdots \oplus E_i^c$  and  $E^{cu,i} = E_i^c \oplus \cdots \oplus E_k^c \oplus E^u$  and consider their extensions  $\tilde{E}^{cs,i}$  and  $\tilde{E}^{cu,i}$  to a small neighborhood of  $\Lambda$ . Then for any  $\epsilon > 0$  there exist constants  $R > r > r_1 > 0$  such that for any  $x \in \Lambda$ , the neighborhood  $B(x, r)$  is foliated by foliations  $\widehat{W}^u(x), \widehat{W}^s(x), \widehat{W}^{cs,i}(x)$  and  $\widehat{W}^{cu,i}(x)$  ( $i = 1, \dots, k$ ) such that for each  $\sigma \in \{u, s, (cs, i), (cu, i)\}$  the following properties hold.

- (a) Almost tangency of invariant distributions. For each  $y \in B(x, r)$ , the leaf  $\widehat{W}_x^\sigma(y)$  is  $C^1$ , and the tangent space  $T_y \widehat{W}_x^\sigma(y)$  lies in a cone of radius  $\epsilon$  about  $\tilde{E}^\sigma(y)$ .
- (b) Coherence.  $\widehat{W}_x^s$  subfoliates  $\widehat{W}_x^{cs,i}$  and  $\widehat{W}_x^u$  subfoliates  $\widehat{W}_x^{cu,i}$  for each  $i \in \{1, \dots, k\}$ .
- (c) Local invariance. For each  $y \in B(x, r)$  we have  $f(\widehat{W}_x^\sigma(y, r_1)) \subset \widehat{W}_{f(x)}^\sigma(f(y))$  and  $f^{-1}(\widehat{W}_x^\sigma(y, r_1)) \subset \widehat{W}_{f^{-1}(x)}^\sigma(f^{-1}(y))$ , where  $\widehat{W}_x^\sigma(y, r_1)$  is the connected components of  $\widehat{W}_x^\sigma(y) \cap B(y, r_1)$  containing  $y$ .
- (d) Uniqueness.  $\widehat{W}_x^s(x) = W^s(x, r)$  and  $\widehat{W}_x^u(x) = W^u(x, r)$ .

We say that a diffeomorphism  $f$  has a *homoclinic tangency* if there is a hyperbolic periodic point  $p$  whose invariant manifolds  $W^s(p)$  and  $W^u(p)$  have a non-transverse intersection. The set of  $C^1$  diffeomorphisms that have some homoclinic tangencies will be denoted  $\mathcal{HT}$ . For a homoclinic tangency, Pacifico and Vieitez [20] proved that surface diffeomorphisms presenting homoclinic tangencies can be  $C^1$ -approximated by non-measure expansive diffeomorphisms. From the result, Lee [14] proved that if  $f$  has a homoclinic tangency associated to a hyperbolic periodic point  $p$ , then there is a  $g$   $C^1$ -close to  $f$  such that  $g$  is not continuum-wise expansive.

**Proposition 1.5** [7, Theorem 1.1] *The diffeomorphism  $f$  in a dense  $\mathcal{G}_\delta$  subset  $\mathcal{G} \subset \text{Diff}(M) \setminus \overline{\mathcal{HT}}$  has the following properties.*

- (a) *Any aperiodic class  $\mathcal{C}$  is partially hyperbolic with a one-dimensional central bundle. Moreover, the Lyapunov exponent along  $E^c$  of any invariant measure supported on  $\mathcal{C}$  is zero.*
- (b) *Any homoclinic class  $H_f(p)$  has a partially hyperbolic structure*

$$T_{H_f(p)}M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u.$$

*Moreover, the minimal stable dimension of the periodic orbits of  $H_f(p)$  is  $\dim E^s$  or  $\dim E^s + 1$ . Similarly, the maximal stable dimension of the periodic orbits of  $H_f(p)$  is  $\dim E^s + k$  or  $\dim E^s + k - 1$ . For every  $i = 1, \dots, k$ , there exist periodic points in  $H_f(p)$  whose Lyapunov exponent along  $E_i^c$  is arbitrarily close to 0. In particular, if  $f \in \mathcal{G}$ , then  $f$  is partially hyperbolic.*

Recently, Pacifico and Vieitez [20] proved that there is a residual subset  $\mathcal{G}$  of  $\text{Diff}(M) \setminus \overline{\mathcal{HT}}$  such that for any Borel probability measure  $\mu$  absolutely continuous with respect to Lebesgue,  $f$  is  $\mu$ -expansive. Lee [15] showed that there is a partially hyperbolic diffeomorphism such that it is not measure expansive. From the facts, we consider continuum-wise expansive for partially hyperbolic diffeomorphisms. The set  $\Lambda$  is *transitive* if there is a point  $x \in \Lambda$  such that  $\omega(x) = \Lambda$ , where  $\omega(x)$  the omega limit set of  $f$ . we say that the set  $\Lambda$  is *robustly transitive* if there are a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  and a neighborhood  $U$  of  $\Lambda$  such that for any  $g \in \mathcal{U}(f)$ ,  $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$  is transitive, where  $\Lambda_g(U)$  is the continuation of  $\Lambda$ . Let  $M = \mathbb{T}^3$  and let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a diffeomorphism.

**Theorem B** *There is a  $C^1$  neighborhood  $\mathcal{U}(f)$  of  $f \in \mathcal{RT}(\mathbb{T}^3)$  and a residual set  $\mathcal{R} \subset \mathcal{U}(f)$  such that for any  $g \in \mathcal{R}$ ,  $g$  is not continuum-wise expansive, where  $\mathcal{RT}(\mathbb{T}^3)$  is the set of all robustly transitive diffeomorphisms on  $\mathbb{T}^3$ .*

## 2 Proof of Theorems

### 2.1 Proof of Theorem A

Let  $M$  be as before, and let  $f \in \text{Diff}(M)$ . The following Franks' lemma [9] will play essential roles in our proofs.

**Lemma 2.1** *Let  $\mathcal{U}(f)$  be any given  $C^1$  neighborhood of  $f$ . Then there exist  $\epsilon > 0$  and a  $C^1$  neighborhood  $\mathcal{U}_0(f) \subset \mathcal{U}(f)$  of  $f$  such that for given  $g \in \mathcal{U}_0(f)$ , a finite*

set  $\{x_1, x_2, \dots, x_N\}$ , a neighborhood  $U$  of  $\{x_1, x_2, \dots, x_N\}$  and linear maps  $L_i : T_{x_i}M \rightarrow T_{g(x_i)}M$  satisfying  $\|L_i - D_{x_i}g\| \leq \epsilon$  for all  $1 \leq i \leq N$ , there exists  $\hat{g} \in \mathcal{U}(f)$  such that  $\hat{g}(x) = g(x)$  if  $x \in \{x_1, x_2, \dots, x_N\} \cup (M \setminus U)$  and  $D_{x_i}\hat{g} = L_i$  for all  $1 \leq i \leq N$ .

The following was proved by [15]. For convenience, we give the proof in the section.

**Lemma 2.2** *If  $f \in \text{Diff}(M)$  has a non-hyperbolic periodic point, then for any neighborhood  $\mathcal{U}(f)$  of  $f$  and any  $\eta > 0$ , there are  $g \in \mathcal{U}(f)$  and a curve  $\gamma$  with the following property:*

1.  $\gamma$  is  $g$  periodic, that is, there is  $n \in \mathbb{Z}$  such that  $g^n(\gamma) = \gamma$ ;
2. the length of  $g^i(\gamma)$  is less than  $\eta$ , for all  $i \in \mathbb{Z}$ ;
3.  $\gamma$  is normally hyperbolic with respect to  $g$ .

**Proof.** Let  $\mathcal{U}(f)$  be a  $C^1$  neighborhood of  $f$ . Suppose  $p$  is not hyperbolic periodic point of  $f$ . For simplicity, we may assume that  $p$  is a fixed point of  $f$ . By Lemma 2.1, there is  $g \in \mathcal{U}(f)$  such that  $D_p g^{\pi(p)}$  has an eigenvalue  $\lambda$  with  $|\lambda| = 1$ . Then  $g(p) = p_g$  and  $T_{p_g}M = E_{p_g}^s \oplus E_{p_g}^c \oplus E_{p_g}^u$ , where  $E_{p_g}^s$  is the eigenspace corresponding to the eigenvalues with modulus less than 1,  $E_{p_g}^u$  is the eigenspace corresponding to the eigenvalues with modulus more than 1, and  $E_{p_g}^c$  is the eigenspace corresponding to  $\lambda$ . If  $\lambda \in \mathbb{R}$  then  $\dim E_{p_g}^c = 1$  and if  $\lambda \in \mathbb{C}$  then  $\dim E_{p_g}^c = 2$ .

In the proof, we consider  $\dim E_{p_g}^c = 1$ . For case  $\dim E_{p_g}^c = 2$ , we can obtain the result as in the case  $\dim E_{p_g}^c = 1$ .

Since  $\dim E_{p_g}^c = 1$  we assume that  $\lambda = 1$ . By Lemma 2.1, there are  $\epsilon > 0$  and  $h \in \mathcal{U}(f)$  such that

- $h(p_g) = g(p_g) = p_g$ ,
- $h(x) = \exp_{p_g} \circ D_{p_g}g \circ \exp_{p_g}^{-1}(x)$  if  $x \in B_\epsilon(p_g)$ , and
- $h(x) = g(x)$  if  $x \notin B_{4\epsilon}(p_g)$ .

Since  $\lambda = 1$ , we can construct a closed small arc  $\mathcal{I}_{p_g} \subset B_\epsilon(p_g) \cap \exp_{p_g}(E_{p_g}^c(\epsilon))$  with its center at  $p_g$  such that

- $\text{diam} \mathcal{I}_{p_g} = \epsilon/4$ ,
- $h(\mathcal{I}_{p_g}) = \mathcal{I}_{p_g}$ , and
- $h|_{\mathcal{I}_{p_g}}$  is the identity map.

Here  $E_{p_g}^c(\epsilon)$  is the  $\epsilon$ -ball in  $E_{p_g}^c$  centered at the origin  $\vec{o}_p$ . Then  $\mathcal{I}_{p_g}$  is normally hyperbolic with respect to  $h$ , and for any  $\eta < \epsilon/4$ , the length of  $\mathcal{I}_{p_g}$  is less than  $\eta$ .  $\square$

By the persistency of normally hyperbolic, we know that there is a neighborhood  $\mathcal{U}(g)$  of  $g$  such that for any  $\tilde{g} \in \mathcal{U}(g)$  there is a curve  $\tilde{\gamma}$  close to  $\gamma$  such that all properties of  $\gamma$  listed in the Lemma 2.2 is also satisfied for  $\tilde{\gamma}$ .

For  $f \in \text{Diff}(M)$ , we say that  $f$  is the *star diffeomorphism* (or  $f$  satisfies the *star condition*) if there is a  $C^1$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that all periodic points of  $g \in \mathcal{U}(f)$  are hyperbolic. Denote by  $\mathcal{F}(M)$  the set of all star diffeomorphisms. Aoki [2] and Hayashi [11] showed that for any dimension case, if  $f \in \mathcal{F}(M)$  then  $f$  is Axiom A without cycles.

**Lemma 2.3** [15, Lemma 3.4] *There is a residual set  $\mathcal{G} \subset \text{Diff}(M)$  such that for any  $f \in \mathcal{G}$ ,*

- *either  $f$  is a star,*
- *or for any  $\varepsilon > 0$  there is a periodic curve  $\gamma$  such that the length of  $f^n(\gamma)$  is less than  $\varepsilon$ , for any  $n \in \mathbb{Z}$ .*

**Proof.** Let  $\mathcal{H}_n$  be the set of  $C^1$  diffeomorphisms  $f$  such that  $f$  has a normally hyperbolic  $\gamma$  which is  $1/n$ -simply periodic curve. Since  $\gamma$  is normally hyperbolic, we know that  $\mathcal{H}_n$  is open. Let  $\mathcal{N}_n = \text{Diff}(M) \setminus \overline{\mathcal{H}_n}$ . Then  $\mathcal{H}_n(\eta) \cup \mathcal{N}_n(\eta)$  is open and dense in  $\text{Diff}(M)$ . Let  $\mathcal{G} = \bigcap_{n \in \mathbb{N}^+} (\mathcal{H}_n \cup \mathcal{N}_n)$ . Then  $\mathcal{G}$  is  $C^1$  residual in  $\text{Diff}(M)$ . Let  $f \in \mathcal{G}$  and assume  $f$  is not a star diffeomorphism, we know that  $f \in \overline{\mathcal{H}_n}$  for any  $n \in \mathbb{N}^+$  by Lemma 2.2. Hence  $f \notin \mathcal{N}_n$  and  $f \in \mathcal{H}_n$  for any  $n$ . We know that  $f$  has a normally hyperbolic  $\gamma$  which is  $\varepsilon$ -simply periodic curve, for any  $\varepsilon > 0$ .  $\square$

The following was proved by [5, Lemma 2.2].

**Lemma 2.4** *Let  $C \subset M$  be a continuum.  $f$  is continuum-wise expansive if and only if there is  $\delta > 0$  such that for all  $x \in M$ , if a continuum  $C \subset \Gamma_\delta(x)$  then  $C$  is a singleton.*

**Proof of Theorem A.** Let  $f \in \mathcal{G}$  be continuum-wise expansive. Suppose by contradiction that  $f \notin \mathcal{F}(M)$ . From Lemma 2.3, for any  $\varepsilon > 0$  there is a periodic curve  $\gamma$  such that the length of  $f^i(\gamma)$  is less than  $\varepsilon$ , for any  $i \in \mathbb{Z}$ . Let  $\Gamma_\epsilon(x) = \{x \in M : d(f^i(x), f^i(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z}\}$ . Since  $f^n(\gamma) = \gamma$  for some  $n \in \mathbb{Z}$ , we know  $\gamma \subset \Gamma_\epsilon(x)$ . By Lemma 2.4,  $\gamma$  should be a singleton which is a contradiction since  $\gamma$  is a nontrivial continuum. Thus if  $f \in \mathcal{G}$  is continuum-wise expansive then it is Axiom A without cycles.  $\square$



## 2.2 Proof of Theorem B

In this section, let  $M = \mathbb{T}^3$  and let  $f : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  be a diffeomorphism. In [17, Theorem B], Mañé constructed a robustly nonhyperbolic transitive diffeomorphism  $f \in \text{Diff}(\mathbb{T}^3)$ . By [8, Theorem B], every robustly transitive diffeomorphism  $f$  on  $\mathbb{T}^3$  is partially hyperbolic. Thus we can find a partially hyperbolic diffeomorphism  $f$  on  $\mathbb{T}^3$  such that  $f$  is robustly nonhyperbolic transitive.

**Remark 2.5** *There is a partially hyperbolic diffeomorphism  $f$  on  $\mathbb{T}^3$  such that  $f$  is robustly nonhyperbolic transitive.*

**Lemma 2.6** [8, Corollary D] and [1, Theorem 4.10] *There is a residual set  $\mathcal{G}_1 \subset \mathcal{RT}(\mathbb{T}^3)$  such that for any  $f \in \mathcal{G}_1$ ,  $f$  is strongly partially hyperbolic, and  $\mathbb{T}^3$  is the homoclinic class  $H_f(p)$ , for some hyperbolic periodic point  $p$ .*

Let  $M$  be a closed smooth  $n(\geq 2)$ -dimensional manifold, and let  $f : M \rightarrow M$  be a diffeomorphism. The following notion was introduced by Yang and Gan [24]. For any  $\epsilon > 0$ , a  $C^1$  curve  $\eta$  is called a  $\epsilon$ -*simply periodic curve* of  $f$  if

- (i)  $\eta$  is diffeomorphic to  $[0, 1]$  and its two endpoints are hyperbolic periodic points of  $f$ ,
- (ii)  $\eta$  is periodic with period  $\pi(\eta)$  and  $L(f^i(\eta)) < \epsilon$  for any  $i \in \{1, 2, \dots, \pi(\eta)\}$ , where  $L(\eta)$  denotes the length of  $\eta$ , and
- (iii)  $\eta$  is normally hyperbolic.

**Lemma 2.7** [24, Lemma 2.1] *There is a residual set  $\mathcal{G}_2 \subset \text{Diff}(M)$  such that for any  $f \in \mathcal{G}_2$ , and any hyperbolic periodic point  $p$  of  $f$ , we have the following:*

*for any  $\epsilon > 0$ , if for any  $C^1$  neighborhood  $\mathcal{U}(f)$  of  $f$  some  $g \in \mathcal{U}(f)$  has a  $\epsilon$ -simply periodic curve  $\eta$  such that two endpoints of  $\eta$  are homoclinically related with  $p_g$  then  $f$  has an  $2\epsilon$ -simply periodic curve  $\zeta$  such that the two endpoints of  $\zeta$  are homoclinically related to  $p$ .*

**Proof of Theorem B.** Let  $\mathcal{U}(f)$  be a  $C^1$  neighborhood of  $f \in \mathcal{RT}(\mathbb{T}^3)$  and let  $f \in \mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$ . Let  $p$  be a hyperbolic periodic point of  $f$ . Then for any  $g \in \mathcal{G} \cap \mathcal{U}(f)$ , we have  $\mathbb{T}^3 = H_g(p_g)$ , where  $p_g$  is the continuation of  $p$ . Since  $H_g(p_g)$  is not hyperbolic, from [23, section 4], for any  $\epsilon > 0$ , there is  $g_1 \in \mathcal{G} \cap \mathcal{U}(f)$  such that  $g_1$  has an  $\epsilon$ -simply periodic curve  $\eta$ , whose endpoints are homoclinically related to  $p_{g_1}$ . Note that  $\eta$  is a closed and connected set, so it is a nontrivial continuum. Let  $e = 2\epsilon$  be



an expansive constant of  $g_1$ . For  $x \in \mathbb{T}^3$ , we set  $\Gamma_e(x) = \{y \in \mathbb{T}^3 : d(g_1^i(x), g_1^i(y)) \leq e \text{ for all } i \in \mathbb{Z}\}$ . Since  $g_1^{\pi(p_g)}(\eta) = \eta$ , we have

$$L(g_1^{i\pi(p_g)}(\eta)) = L(\eta) < e.$$

Clearly  $\eta \subset \Gamma_e(x)$ . Since  $\eta$  is compact and connected, and so,  $\eta$  is not singleton which is a contradiction.  $\square$

## Acknowledgement

This work is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (No-2014R1A1A1A05002124).

## References

- [1] F. Abdenur, C. Bonatti and S. Crovisier, *Nonuniform hyperbolicity for  $C^1$ -generic diffeomorphisms*, Israel J. Math., **183**(2011), 1-60.
- [2] N. Aoki, *The set of Axiom A diffeomorphisms with no cycles*. Bol. Soc. Bras. Mat., **23**(1992), 21-65.
- [3] A. Arbieto, *Periodic orbits and expansiveness*, Math. Z., **269**(2011), 801-807.
- [4] A. Artigue, *Robustly  $N$ -expansive surface diffeomorphisms*, Disc. Contin. Dynam. Syst. **36** (2016), 2367-2376.
- [5] A. Artigue and D. Carrasco-Olivera, *A note on measure expansive diffeomorphisms*, J. Math. Anal. Appl., **428**(2015), 713-716.
- [6] K. Burns and A. Wilkinson, *On the ergodicity of partially hyperbolic systems*, Ann. of Math. **171**(2010), 451-489.
- [7] S. Crovisier, M. Sambarino and D. Yang, *Partial hyperbolicity and homoclinic tangencies*, J. Eur. Math. Soc, **17**(2015), 1-49.
- [8] L. J. Díaz, E. R. Pujals and R. Ures, *Partial hyperbolicity and robust transitivity*, Acta Math., **183**(1999), 1-43.
- [9] J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301–308.

- [10] J. Franks and C. Robinson, *A quasi-Anosov diffeomorphism that is not Anosov* Trans. Amer. Math. Soc, **223** (1976), 267-278.
- [11] S. Hayashi, *Diffeomorphisms in  $\mathcal{F}^1(M)$  satisfy Axiom A*, Ergodic Thoery & Dynam. Syst., **12** (1992), 233-253.
- [12] H. Kato, *Continuum-wise expansive homeomorphisms*, Can. J. Math., **45**(1993), 576-598.
- [13] M. Lee, *General Expansiveness for Diffeomorphisms from the Robust and Generic Properties*, to appear in J. Dynam. Cont. Syst.
- [14] M. Lee, *Continuum-wise expansive homoclinic classes for generic diffeomorphisms*, Publ. Math. Debrecen **88**(2016), 193-200.
- [15] M. Lee, *Measure expansiveness for generic diffeomorphisms*, preprint.
- [16] R. Mañé, *Expansive diffeomorphisms*, Lecture Notes in Math. **468**, Springer, Berlin 1975.
- [17] R. Mañé, *Contributions to the stability conjecture*, Topology, **17**(1978), 383-396.
- [18] C. A. Morales, *A generalization of expansivity*, Disc. Contin. Dynam. Syst. **32** (2012), 293-301.
- [19] C. A. Morales and V. F. Sirvent, *Expansive measures*, 29 Colóquio Brasileiro de Matemática, 2013.
- [20] M. J. Pacifico and J. L. Vieites, *On measure expansive diffeomorphisms*, Proc. Amer. Math. Soc., **143**(2015), 811-819.
- [21] K. Sakai, *Continuum-wise expansive diffeomorphisms*, Publca. Mate. **41**(1997), 375-382.
- [22] K. Sakai, N. Sumi, and K. Yamamoto, *Measure-expansive diffeomorphisms*, J. Math. Anal. Appl. **414** (2014), 546-552.
- [23] M. Sambarino and J. Vieitez, *On  $C^1$ -persistently expansive homoclinic classes*, Discrete Contin. Dynam. Syst., **14**(2006), 465-481.
- [24] D. Yang and S. Gan, *Expansive homoclinic classes*, Nonlinearity, **22**(2009), 729-733.